ADVANCED GCE UNIT **4764/01**

MATHEMATICS (MEI)

Mechanics 4

FRIDAY 22 JUNE 2007

Morning
Time: 1 hour 30 minutes

Additional materials:
Answer booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by g m s^{-2}. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.

ADVICE TO CANDIDATES

- Read each question carefully and make sure you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

Section A (24 marks)

1 A light elastic string has one end fixed to a vertical pole at A. The string passes round a smooth horizontal peg, P, at a distance a from the pole and has a smooth ring of mass m attached at its other end B. The ring is threaded onto the pole below A. The ring is at a distance y below the horizontal level of the peg. This situation is shown in Fig. 1.

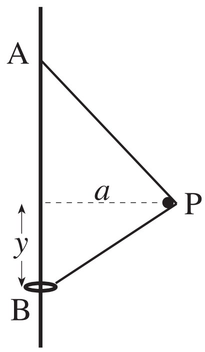

Fig. 1

The string has stiffness k and natural length equal to the distance AP.

(i) Express the extension of the string in terms of y and a. Hence find the potential energy of the system relative to the level of P. [5]

(ii) Use the potential energy to find the equilibrium position of the system, and show that it is stable. [5]

(iii) Calculate the normal reaction exerted by the pole on the ring in the equilibrium position. [2]

2 A railway truck of mass m_0 travels along a horizontal track. There is no driving force and the resistances to motion are negligible. The truck is being filled with coal which falls vertically into it at a mass rate k. The process starts as the truck passes a point O with speed u. After time t, the truck has velocity v and the displacement from O is x.

(i) Show that $v = \dfrac{m_0 u}{m_0 + kt}$ and find x in terms of m_0, u, k and t. [9]

(ii) Find the distance that the truck has travelled when its speed has been halved. [3]

Section B (48 marks)

3 **(i)** Show, by integration, that the moment of inertia of a uniform rod of mass m and length $2a$ about an axis through its centre and perpendicular to the rod is $\frac{1}{3}ma^2$. [6]

A pendulum of length 1 m is made by attaching a uniform sphere of mass 2 kg and radius 0.1 m to the end of a uniform rod AB of mass 1.2 kg and length 0.8 m, as shown in Fig. 3. The centre of the sphere is collinear with A and B.

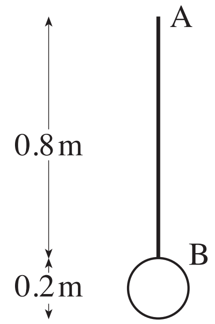

Fig. 3

(ii) Find the moment of inertia of the pendulum about an axis through A perpendicular to the rod. [7]

The pendulum can swing freely in a vertical plane about a fixed horizontal axis through A.

(iii) The pendulum is held with AB at an angle α to the downward vertical and released from rest. At time t, AB is at an angle θ to the vertical. Find an expression for $\dot{\theta}^2$ in terms of θ and α. [6]

(iv) Hence, or otherwise, show that, provided that α is small, the pendulum performs simple harmonic motion. Calculate the period. [5]

4 A particle of mass 2 kg starts from rest at a point O and moves in a horizontal line with velocity $v\,\mathrm{m\,s^{-1}}$ under the action of a force F N, where $F = 2 - 8v^2$. The displacement of the particle from O at time t seconds is x m.

(i) Formulate and solve a differential equation to show that $v^2 = \frac{1}{4}(1 - \mathrm{e}^{-8x})$. [7]

(ii) Hence express F in terms of x and find, by integration, the work done in the first 2 m of the motion. [6]

(iii) Formulate and solve a differential equation to show that $v = \frac{1}{2}\left(\frac{1-\mathrm{e}^{-4t}}{1+\mathrm{e}^{-4t}}\right)$. [7]

(iv) Calculate v when $t = 1$ and when $t = 2$, giving your answers to four significant figures. Hence find the impulse of the force F over the interval $1 \leqslant t \leqslant 2$. [4]

Mark Scheme 4764
June 2007

1(i)	$x = \text{PB}$	M1	May be implied	
	$x = \sqrt{a^2 + y^2}$	A1		
	$V = \frac{1}{2}kx^2 - mgy$	M1	EPE term	
		M1	GPE term	
	$= \frac{1}{2}k\left(a^2 + y^2\right) - mgy$	A1	cao	
				5
(ii)	$\frac{dV}{dy} = ky - mg$	M1	Differentiate their V	
	equilibrium $\Rightarrow \frac{dV}{dy} = 0$	B1	Seen or implied	
	$\Rightarrow y = \frac{mg}{k}$	A1	cao	
	$\frac{d^2V}{dy^2} = k > 0$	M1	Consider sign of V'' (or V' either side)	
	$\Rightarrow$ stable	E1	Complete argument	
				5
(iii)	$R = T \sin P\hat{B}A = k \cdot PB \cdot \frac{a}{PB}$	M1	Use Hooke's law and resolve	
	$= ka$	A1		
				2
2(i)	$\frac{d}{dt}(mv) = 0 \Rightarrow mv$ constant	M1	Or no external forces $\Rightarrow$ momentum conserved, or attempt using δ terms.	
	hence $mv = m_0 u$	A1		
	$\frac{dm}{dt} = k$	B1	$\frac{dm}{dt} = k$ seen	
	$\Rightarrow m = m_0 + kt$	B1	$m_0 + kt$ stated or clearly used as mass	
	$v = \frac{m_0 u}{m} = \frac{m_0 u}{m_0 + kt}$	E1	Complete argument (dependent on all previous marks and $m_0 + kt$ derived, not just stated)	
	$x = \int \frac{m_0 u}{m_0 + kt}\, dt$	M1	Integrate v	
	$= \frac{m_0 u}{k} \ln\left(m_0 + kt\right) + A$	A1	cao	
	$x = 0, t = 0 \Rightarrow A = -\frac{m_0 u}{k} \ln m_0$	M1	Use condition	
	$x = \frac{m_0 u}{k} \ln\left(\frac{m_0 + kt}{m_0}\right)$	A1	cao	
				9
(ii)	$v = \frac{1}{2}u \Rightarrow m_0 + kt = 2m_0$	M1	Attempt to calculate value of m or t	
	$\Rightarrow x = \frac{m_0 u}{k} \ln\left(\frac{2m_0}{m_0}\right)$	M1	Substitute their m or t into x	
	$\Rightarrow x = \frac{m_0 u}{k} \ln 2$	F1	$t = \frac{m_0}{k}$ or $m = 2m_0$ in their x	
				3

3(i)	$I = \int_{-a}^{a} \rho x^2 \, \mathrm{d}x$	M1 A1	Set up integral Or equivalent
	$\rho = \dfrac{m}{2a}$	M1	Use mass per unit length in integral or I
	$I = \dfrac{m}{2a}\left[\tfrac{1}{3}x^3\right]_{-a}^{a}$	M1	Integrate
	$= \tfrac{1}{6}ma^2 - -\tfrac{1}{6}ma^2$	M1	Use limits
	$\tfrac{1}{3}ma^2$	E1	Complete argument
			6
(ii)	$I_{\text{rod}} = \tfrac{1}{3}\times 1.2 \times 0.4^2 + 1.2 \times 0.4^2$	M1	Use $\tfrac{1}{3}ma^2$ or $\tfrac{4}{3}ma^2$
		A1	Rod term(s) all correct
	$I_{\text{sphere}} = \tfrac{2}{5}\times 2 \times 0.1^2 + 2 \times 0.9^2$	M1 M1 A1	Use formula for sphere Use parallel axis theorem Sphere terms all correct
	$I = I_{\text{rod}} + I_{\text{sphere}} = 1.884$	M1 A1	Add moment of inertia for rod and sphere cao
			7
(iii)	$\tfrac{1}{2}I\dot{\theta}^2 - 1.2g \times 0.4\cos\theta - 2g \times 0.9\cos\theta$ $= -1.2g \times 0.4\cos\alpha - 2g \times 0.9\cos\alpha$	M1 M1 M1 A1	Use energy KE term Reasonable attempt at GPE terms All terms correct (but ignore signs)
	$\dot{\theta}^2 = \dfrac{4.56g}{1.884}(\cos\theta - \cos\alpha)$	M1 F1	Rearrange Only follow an incorrect I
			6
(iv)	$2\dot{\theta}\ddot{\theta} = \dfrac{4.56g}{1.884}(-\sin\theta\,\dot{\theta})$ or $I\ddot{\theta} = -1.2g \times 0.4\sin\theta - 2g \times 0.9\sin\theta$	M1	Differentiate, or use moment $= I\ddot{\theta}$
		F1	Equation for $\ddot{\theta}$ (only follow their I or $\dot{\theta}^2$)
	$\sin\theta \approx \theta \Rightarrow \ddot{\theta} = -11.86\theta$	M1	Use small angle approximation (in terms of θ)
	i.e. SHM	E1	All correct (for their I) and make conclusion
	$T \approx \dfrac{2\pi}{\sqrt{11.86}} \approx 1.82$	F1	$\dfrac{2\pi}{\text{their } \omega}$
			5

4(i)	$2v\dfrac{dv}{dx} = 2 - 8v^2$	M1 A1	N2L	
	$\int \dfrac{v}{1-4v^2}\,dv = \int dx$	M1	Separate	
	$-\frac{1}{8}\ln\left\|1-4v^2\right\| = x + c_1$	A1	LHS	
	$x = 0, v = 0 \Rightarrow c_1 = 0$	M1	Use condition	
	$1 - 4v^2 = e^{-8x}$	M1	Rearrange	
	$v^2 = \frac{1}{4}\left(1 - e^{-8x}\right)$	E1	Complete argument	
				7
(ii)	$F = 2 - 8v^2 = 2 - 2\left(1 - e^{-8x}\right)$	M1	Substitute given v^2 into F	
	$= 2e^{-8x}$	A1	cao	
	Work done = $\int_0^2 F\,dx$	M1	Set up integral of F	
	$= \int_0^2 2e^{-8x}\,dx$	A1	cao	
	$= \left[-\frac{1}{4}e^{-8x}\right]_0^2$	M1	Integrate	
	$= \frac{1}{4}\left(1 - e^{-16}\right)$	A1	Accept $\frac{1}{4}$ or 0.25 from correct working	
				6
(iii)	$2\dfrac{dv}{dt} = 2 - 8v^2$	M1	N2L	
	$\frac{1}{4}\int \dfrac{1}{\frac{1}{4} - v^2}\,dv = \int dt$	M1	Separate	
	$\frac{1}{4}\ln\left\|\dfrac{\frac{1}{2}+v}{\frac{1}{2}-v}\right\| = t + c_2$	A1	LHS	
	$t = 0, v = 0 \Rightarrow c_2 = 0$	M1	Use condition	
	$\dfrac{\frac{1}{2}+v}{\frac{1}{2}-v} = e^{4t}$	M1	Rearrange (remove log)	
	$1 + 2v = e^{4t}(1 - 2v)$			
	$2v\left(1 + e^{4t}\right) = e^{4t} - 1$	M1	Rearrange (v in terms of t)	
	$v = \frac{1}{2}\left(\dfrac{e^{4t}-1}{e^{4t}+1}\right) = \frac{1}{2}\left(\dfrac{1-e^{-4t}}{1+e^{-4t}}\right)$	E1	Complete argument	
				7
(v)	$t = 1 \Rightarrow v = 0.4820$ $t = 2 \Rightarrow v = 0.4997$ Impulse $= mv_2 - mv_1$ = 0.0353	B1 B1 M1 A1	 Use impulse-momentum equation Accept anything in interval [0.035, 0.036]	
				4

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General Comments

The standard of work was very high with most candidates demonstrating a good grasp of the mechanics and sound algebraic skills.

Comments on Individual Questions

1 **Stability**

(i) The calculation of potential energy was almost always done correctly.

(ii) The position of equilibrium was also usually calculated correctly.

(iii) When calculating the reaction, some candidates omitted to resolve the tension in the string. Some candidates made very lengthy calculations, often finding difficulty with the angle.

2 **Variable mass**

(i) Although many candidates were able to find the velocity, many did not show sufficient working for a given answer, in particular the derivation of the expression for mass. Some candidates integrated the Newton's second law equation directly to get mv = constant, others appealed to the conservation of momentum. These methods were perfectly acceptable, but some unnecessarily expanded $\frac{\mathrm{d}}{\mathrm{d}t}(mv)$ and set up and solved a differential equation. Finding the displacement from the given velocity was usually done correctly.

(ii) This was usually done correctly.

3 **Rotation**

(i) This was often done well, but some derivations lacked clarity. Some candidates set up an integral which bore no relation to the mechanics but just happened to give the correct answer.

(ii) This was also often done well, but some did not take account of the position of the axis for the rod or for the sphere.

(iii) Candidates who used energy often gave good solutions to this part. However errors in the potential energy were common. A clear diagram generally was helpful to candidates. Candidates who attempted this via a non-energy method were rarely successful.

(iv) Some candidates successfully used the rotational equation of motion, others successfully differentiated their expression from the previous part. However sign errors were common. Some candidates did not seem to know what was required here. When showing SHM, candidates are expected to make a conclusion once they have derived the relevant equation (simply stating "hence SHM" would be enough).

4 **Variable acceleration**

(i) Most candidates successfully found the required expression, but some omitted the constant of integration when solving the differential equation. It is vital for candidates to realise that the constant must always be included.

(ii) This was usually completed correctly.

(iii) The integration required in this part caused some problems. Some candidates used the standard result from the formula book, as intended. Others used partial fractions and others simply gave up. The constant of integration was sometimes omitted.

(iv) This was often completed correctly, but some candidates did not realise that they could use impulse equals change in momentum.